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Some Fixed Point Theorems And Their Application

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Abstract – In this paper we prove some common fixed point theorem in rational inequality on complete metric spaces. As applications, the existence and uniqueness results of common solution for some functional equations and system of functional equations in Dynamic programming are given.

Key words - common fixed point, complete metric space, common solution, functional equation, system of functional equations, dynamic programming.

1.INTRODUCTION

Contractive type mapping and corresponding fixed or common fixed point theorems and their applications given by many researchers. Bellman [6] studied the existence of solutions for some classes of functional equation arising in Dynamic programming. Bellman and Lee [7] gives the basic form of functional equations in Dynamic programming as follows:

$$f(x) = \underset{y \in D}{\mathrm{opt}} H\{x, y, f(T(x, y))\}, \ \forall x \in S \qquad (1.1)$$

Where opt represents sup or inf, x and y denotes the state and decision vectors respectively, T stands for the transformation of the process and f(x) represents the optimal return function with the initial state x. After Bellman, many researcher Baskaran and Subrahmanyam [8], Bhakta and Choudhury [4], Bhakta and Mitra [5], Chang and Ma [9], Liu [11]-[14], Liu, Agarwal, and Kang [15], Liu and Ume [19], Pathak and Fisher [3], Zhang [10] investigate the existence and uniqueness of solution and common solution for some kinds of functional equations and systems of functional equations.

Ray [1] established two common fixed point theorems for the following self mapping f, g, and h in a complete metric space(X, d):

$$d(fx, gy) \le d(hx, hy) - w(d(hx, hy)) \quad \forall x, y \in X$$

Liu [11], introduced and studied a class contractive type mapping as:

$$d(fx, gy) \le \max\{d(hx, hy), d(hx, fx), d(hy, gy)\} - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\}) \quad \forall x, y \in X$$

Z. Liu, L. Wang, H. K. Kim and S. M. Kang [22], proved some common fixed point theorems for contractive type mappings for three self mapping as follows –

$$d(fx, gy) \le \max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)], \frac{d(hx, fx)d(hy, gy)}{1 + d(fx, gy)}, \frac{d(hx, fx)d(hy, gy)}{1 + d(hx, hy)}\} - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)], \frac{d(hx, fx)d(hy, gy)}{1 + d(fx, gy)}, \frac{d(hx, fx)d(hy, gy)}{1 + d(hx, hy)}\}), \quad \forall x, y \in X.$$

J. Li, M. Fu, Z. Liu and S. M. Kang [21] proved some common fixed point theorem for contractive type mappings as follows:

$$d(fx, gy) \le \max\{d(fx, tx), d(gy, hy), d(hy, tx), \frac{1}{2}[d(fx, hy) + d(gy, tx)], \frac{d(fx, tx)d(gy, hy)}{1 + d(hy, tx)}, \frac{d(fx, hy)d(gy, tx)}{1 + d(hy, tx)}, \frac{d(fx, hy)d(gy, tx)}{1 + d(fx, gy)}\} - W(\max\{d(fx, tx), d(gy, hy), d(hy, tx), \frac{1}{2}[d(fx, hy) + d(gy, tx)], \frac{d(fx, tx)d(gy, hy)}{1 + d(hy, tx)}, \frac{d(fx, hy)d(gy, tx)}{1 + d(hy, tx)}, \frac{d(fx, hy)d(gy, tx)}{1 + d(fx, gy)}\}), \forall x, y \in X$$

In this paper we give some sufficient conditions for existence and uniqueness of common fixed point on complete metric space (X, d):

$$d(fx,gy) \le \max\Big\{\frac{d(fx,tx).d(gy,hy)}{d(fx,hy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,gy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,hy)+d(fx,tx)},\frac{d(fx,tx).d(hy,tx)}{d(gy,hy)+d(gy,tx)},\frac{d(gy,hy).d(hy,tx)}{d(fx,gy)+d(fx,hy)}\Big\}$$
$$-W\Big(\max\Big\{\frac{d(fx,tx).d(gy,hy}{d(fx,hy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,gy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,hy)+d(fx,tx)},\frac{d(fx,tx).d(hy,tx)}{d(gy,hy)+d(gy,tx)},\frac{d(gy,hy).d(hy,tx)}{d(fx,gy)+d(fx,hy)}\Big\}\Big)$$

$\forall\, x,y \in X$, where $\emptyset = \{W: R^+ \to R^+$ is a continuous function such that $0 < W(r) < r$ for all $r \in R^+\{0\}\}$, $R^+ = [0,\infty)$, $R = (-\infty,\infty)$.

2. MAIN RESUTS

Theorem 2.1 Let (X, d) be a complete metric space f, g, h and t be four continuous mapping from X into itself satisfying $ft = tf, gh = hg, f(X) \subseteq h(X)$ and $g(X) \subseteq t(X)$. If there exist $W \in \emptyset$ satisfying,

$$d(fx,gy) \le \max\Big\{\frac{d(fx,tx).d(gy,hy)}{d(fx,hy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,gy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,hy)+d(fx,tx)},\frac{d(fx,tx).d(hy,tx)}{d(gy,hy)+d(gy,tx)},\frac{d(gy,hy).d(hy,tx)}{d(fx,gy)+d(fx,hy)}\Big\}$$
$$-W\Big(\max\Big\{\frac{d(fx,tx).d(gy,hy}{d(fx,hy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,gy)+d(hy,tx)},\frac{d(fx,hy).d(gy,tx)}{d(fx,hy)+d(fx,tx)},\frac{d(fx,tx).d(hy,tx)}{d(gy,hy)+d(gy,tx)},\frac{d(gy,hy).d(hy,tx)}{d(fx,gy)+d(fx,hy)}\Big\}\Big) \quad (2.1)$$

then f, g, h and t have a unique common fixed point in X.

Proof- Let x_0 be an arbitrary point in X.Since $f(X) \subseteq h(X)$ and $g(X) \subseteq t(X)$. It follows that there exist two sequence $\{y_n\}_{n\ge1}$ and $\{x_n\}_{n\ge0}$ such that $y_{2n+1} = fx_{2n} = hx_{2n+1}$ for $n \ge 0$ and $y_{2n} = gx_{2n-1} = tx_{2n}$ for $n \ge 1$. Define $d_n = d(y_n, y_{n+1})$ for $n \ge 1$.
We first show that

$$d_{n+1} \le d_n - W(d_n), \quad \forall\, n \ge 1. \qquad (2.2)$$

Let $n \ge 1$, by eq.(2.1) for $x = x_{2n}$ and $y = x_{2n+1}$,
We have

$$d(fx_{2n},gx_{2n+1}) \le \max\Big\{\frac{d(fx_{2n},tx_{2n}).d(gx_{2n+1},hx_{2n+1})}{d(fx_{2n},hx_{2n+1})+d(hx_{2n+1},tx_{2n})},\frac{d(fx_{2n},hx_{2n+1}).d(gx_{2n+1},tx_{2n})}{d(fx_{2n},gx_{2n+1})+d(hx_{2n+1},tx_{2n})},\frac{d(fx_{2n},hx_{2n+1}).d(gx_{2n+1},tx_{2n})}{d(fx_{2n},hx_{2n+1})+d(fx_{2n},tx_{2n})},\frac{d(fx_{2n},tx_{2n}).d(hx_{2n+1},tx_{2n})}{d(gx_{2n+1},hx_{2n+1})+d(gx_{2n+1},tx_{2n})},\frac{d(gx_{2n+1},hx_{2n+1}).d(hx_{2n+1},tx_{2n})}{d(fx_{2n},gx_{2n+1})+d(fx_{2n},hx_{2n+1})}\Big\}$$
$$-W\Big(\max\Big\{\frac{d(fx_{2n},tx_{2n}).d(gx_{2n+1},hx_{2n+1})}{d(fx_{2n},hx_{2n+1})+d(hx_{2n+1},tx_{2n})},\frac{d(fx_{2n},hx_{2n+1}).d(gx_{2n+1},tx_{2n})}{d(fx_{2n},gx_{2n+1})+d(hx_{2n+1},tx_{2n})},\frac{d(fx_{2n},hx_{2n+1}).d(gx_{2n+1},tx_{2n})}{d(fx_{2n},hx_{2n+1})+d(fx_{2n},tx_{2n})},\frac{d(fx_{2n},tx_{2n}).d(hx_{2n+1},tx_{2n})}{d(gx_{2n+1},hx_{2n+1})+d(gx_{2n+1},tx_{2n})},\frac{d(gx_{2n+1},hx_{2n+1}).d(hx_{2n+1},tx_{2n})}{d(fx_{2n},gx_{2n+1})+d(fx_{2n},hx_{2n+1})}\Big\}\Big)$$

$$= \max\Big\{\frac{d(y_{2n+1},y_{2n}).d(y_{2n+2},y_{2n+1})}{d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n})},\frac{d(y_{2n+1},y_{2n+1}).d(y_{2n+2},y_{2n})}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n})},\frac{d(y_{2n+1},y_{2n+1}).d(y_{2n+2},y_{2n})}{d(fx_{2n},hx_{2n+1})+d(fx_{2n},tx_{2n})},\frac{d(y_{2n+1},y_{2n}).d(y_{2n+1},y_{2n})}{d(y_{2n+2},y_{2n+1})+d(y_{2n+2},y_{2n})},\frac{d(y_{2n+2},y_{2n+1}).d(y_{2n+1},y_{2n})}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})}\Big\}$$
$$-W\Big(\max\Big\{\frac{d(y_{2n+1},y_{2n}).d(y_{2n+2},y_{2n+1})}{d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n})},\frac{d(y_{2n+1},y_{2n+1}).d(y_{2n+2},y_{2n})}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n})},\frac{d(y_{2n+1},y_{2n+1}).d(y_{2n+2},y_{2n})}{d(fx_{2n},hx_{2n+1})+d(fx_{2n},tx_{2n})},\frac{d(y_{2n+1},y_{2n}).d(y_{2n+1},y_{2n})}{d(y_{2n+2},y_{2n+1})+d(y_{2n+2},y_{2n})},\frac{d(y_{2n+2},y_{2n+1}).d(y_{2n+1},y_{2n})}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})}\Big\}\Big)$$

$$d_{2n+1} \le \max\Big\{\frac{d_{2n}.d_{2n+1}}{d_{2n}}, 0, 0, \frac{d_{2n}.d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}, \frac{d_{2n+1}.d_{2n}}{d_{2n+1}}\Big\}$$
$$-W\left(\max\left\{\frac{d_{2n}.d_{2n+1}}{d_{2n}}, 0, 0, \frac{d_{2n}.d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}, \frac{d_{2n+1}.d_{2n}}{d_{2n+1}}\right\}\right)$$
$$= \max\left\{d_{2n+1}, 0, 0, \frac{d_{2n}.d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}, d_{2n}\right\}$$
$$-W\left(\max\left\{d_{2n+1}, 0, 0, \frac{d_{2n}.d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}, d_{2n}\right\}\right)$$
$$= \max\left\{d_{2n+1}, 0, 0, \frac{d_{2n}.d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}, d_{2n}\right\}$$

$$-W(\max\left\{\begin{matrix} d_{2n+1}\,, 0\,, 0, \\ \dfrac{d_{2n}.\,d_{2n}}{d_{2n+1}+d(y_{2n+2},y_{2n})}\,, d_{2n}\end{matrix}\right\})$$

$$= \max\{d_{2n+1}\,, d_{2n}\} - W(\max\{d_{2n+1}\,, d_{2n}\}) \qquad (2.3)$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n \geq 1$ Then from eq. (2.2),

$d_{2n+1} \leq d_{2n+1} - W(d_{2n+1}) < d_{2n+1}$, a contradiction.

Hence we have $d_{2n+1} \leq d_{2n}$

And so $d_{2n+1} \leq d_{2n} - W(d_{2n})$, for all $n \geq 1$.

Consequently we get, $d_{2n} \leq d_{2n-1} - W(d_{2n-1})$ for all $n \geq 1$.

Hence eq. (2.2) holds. Therefore the series of non-negative terms $\sum_{n=1}^{\infty} W(d_n)$ is convergent.

Hence $\lim_{n\to\infty} W(d_n) = 0$, Since $\{d_n\}_{n\geq 1}$ is a non-negative decreasing sequence, it converges to some point p. By the continuity of W we have,

$$W(p) = \lim_{n\to\infty} W(d_n) = 0,$$

Which means that $p = 0$.Hence $\lim_{n\to\infty} d_n = 0$.

Now next we show that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Thus there exists a positive number ϵ such that for each even integer 2k, there are even integer 2m(k) and 2n(k) such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \qquad 2m(k) > 2n(k) > 2k.$$

for each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}) \leq \epsilon\,,\ d(y_{2m(k)}, y_{2n(k)}) > \epsilon \qquad (2.4)$$

It follows that for each even integer 2k,

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}$$

Using eq. (2.4) and above inequality we deduce that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon \qquad (2.5)$$

By the triangle inequality we get for each even integer 2k

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leq d_{2m(k)-1}$$

$$\left|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leq d_{2m(k)-1} + d_{2n(k)}$$

and

$$\left|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2m(k)}, y_{2n(k)})\right| \leq d_{2n(k)}$$

from eq.(2.5), we get

$$\epsilon = \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)})$$

Again from the eq. (2.1)

$$d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(fx_{2n(k)}, gx_{2m(k)-1})$$

$$\leq d_{2n(k)} + \max\{\frac{d(fx_{2n(k)}, tx_{2n(k)}).\,d(gx_{2m(k)-1}, hx_{2m(k)-1})}{d(fx_{2n(k)}, hx_{2m(k)-1}) + d(hx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, hx_{2m(k)-1}).\,d(gx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, gx_{2m(k)-1}) + d(hx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, hx_{2m(k)-1}).\,d(gx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, hx_{2m(k)-1}) + d(fx_{2n(k)}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, tx_{2n(k)}).\,d(hx_{2m(k)-1}, tx_{2n(k)})}{d(gx_{2m(k)-1}, hx_{2m(k)-1}) + d(gx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(gx_{2m(k)-1}, hx_{2m(k)-1}).\,d(hx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, gx_{2m(k)-1}) + d(fx_{2n(k)}, hx_{2m(k)-1})}\}$$

$$-W(\max\{\frac{d(fx_{2n(k)}, tx_{2n(k)}).\,d(gx_{2m(k)-1}, hx_{2m(k)-1})}{d(fx_{2n(k)}, hx_{2m(k)-1}) + d(hx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, hx_{2m(k)-1}).\,d(gx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, gx_{2m(k)-1}) + d(hx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, hx_{2m(k)-1}).\,d(gx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, hx_{2m(k)-1}) + d(fx_{2n(k)}, tx_{2n(k)})},$$

$$\frac{d(fx_{2n(k)}, tx_{2n(k)}).\,d(hx_{2m(k)-1}, tx_{2n(k)})}{d(gx_{2m(k)-1}, hx_{2m(k)-1}) + d(gx_{2m(k)-1}, tx_{2n(k)})},$$

$$\frac{d(gx_{2m(k)-1}, hx_{2m(k)-1}).\,d(hx_{2m(k)-1}, tx_{2n(k)})}{d(fx_{2n(k)}, gx_{2m(k)-1}) + d(fx_{2n(k)}, hx_{2m(k)-1})}\}$$

$$= \max\{\frac{d(y_{2n(k)+1}, y_{2n(k)}).\,d(y_{2m(k)}, y_{2m(k)-1})}{d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2m(k)-1}).\,d(y_{2m(k)}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)}) + d(y_{2m(k)-1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2m(k)-1}).\,d(y_{2m(k)}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2n(k)}).\,d(y_{2m(k)-1}, y_{2n(k)})}{d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)})},$$

$$\frac{d(y_{2m(k)}, y_{2m(k)-1}).\,d(y_{2m(k)-1}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})}\}$$

$$-W(\max\{\frac{d(y_{2n(k)+1}, y_{2n(k)}).\,d(y_{2m(k)}, y_{2m(k)-1})}{d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2m(k)-1}).\,d(y_{2m(k)}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)}) + d(y_{2m(k)-1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2m(k)-1}).\,d(y_{2m(k)}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2n(k)})},$$

$$\frac{d(y_{2n(k)+1}, y_{2n(k)}).\,d(y_{2m(k)-1}, y_{2n(k)})}{d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)})},$$

$$\frac{d(y_{2m(k)}, y_{2m(k)-1}).\,d(y_{2m(k)-1}, y_{2n(k)})}{d(y_{2n(k)+1}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})}\})$$

as $k \to \infty$,we infer that

$$\epsilon \leq \max\left\{0, \frac{\epsilon^2}{2\epsilon}, \frac{\epsilon^2}{\epsilon}, 0, 0\right\} - W\left(\max\left\{0, \frac{\epsilon^2}{2\epsilon}, \frac{\epsilon^2}{\epsilon}, 0, 0\right\}\right)$$

$$\epsilon \leq \epsilon - W(\epsilon)$$

From the above we have $W(\epsilon) \leq 0$. This is a contradiction. Thus $\{y_n\}_{n\geq 1}$ is a Cauchy sequence. Therefore $\{y_n\}_{n\geq 1}$ converges to a point $z \in X$ by completeness of X.

It follows that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} tx_{2n} = z$$

By the continuity of h, f, t and g, and $ft = tf, hg = gh$,

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Now for any $n \ge 0$,

$$d(tfx_{2n},hgx_{2n+1}) = d(ftx_{2n},ghx_{2n+1})$$

$$\le \max\{\frac{d(ftx_{2n},ttx_{2n}).d(ghx_{2n+1},hhx_{2n+1})}{d(ftx_{2n},hhx_{2n+1})+d(hhx_{2n+1},ttx_{2n})}, \frac{d(ftx_{2n},hhx_{2n+1}).d(ghx_{2n+1},ttx_{2n})}{d(ftx_{2n},ghx_{2n+1})+d(hhx_{2n+1},ttx_{2n})}, \frac{d(ftx_{2n},hhx_{2n+1}).d(ghx_{2n+1},ttx_{2n})}{d(ftx_{2n},hhx_{2n+1})+d(ftx_{2n},ttx_{2n})}, \frac{d(ftx_{2n},ttx_{2n}).d(hhx_{2n+1},ttx_{2n})}{d(ghx_{2n+1},hhx_{2n+1})+d(ghx_{2n+1},ttx_{2n})}, \frac{d(ghx_{2n+1},hhx_{2n+1}).d(hhx_{2n+1},ttx_{2n})}{d(ftx_{2n},ghx_{2n+1})+d(ftx_{2n},hhx_{2n+1})}\}$$

$$-W(\max\{\frac{d(ftx_{2n},ttx_{2n}).d(ghx_{2n+1},hhx_{2n+1})}{d(ftx_{2n},hhx_{2n+1})+d(hhx_{2n+1},ttx_{2n})}, \frac{d(ftx_{2n},hhx_{2n+1}).d(ghx_{2n+1},ttx_{2n})}{d(ftx_{2n},ghx_{2n+1})+d(hhx_{2n+1},ttx_{2n})}, \frac{d(ftx_{2n},hhx_{2n+1}).d(ghx_{2n+1},ttx_{2n})}{d(ftx_{2n},hhx_{2n+1})+d(ftx_{2n},ttx_{2n})}, \frac{d(ftx_{2n},ttx_{2n}).d(hhx_{2n+1},ttx_{2n})}{d(ghx_{2n+1},hhx_{2n+1})+d(ghx_{2n+1},ttx_{2n})}, \frac{d(ghx_{2n+1},hhx_{2n+1}).d(hhx_{2n+1},ttx_{2n})}{d(ftx_{2n},ghx_{2n+1})+d(ftx_{2n},hhx_{2n+1})}\})$$

As $n \to \infty$, we get

$$d(tz,hz)$$

$$\le \max\{\frac{d(fz,tz).d(gz,hz)}{d(fz,hz)+d(hz,tz)}, \frac{d(fz,hz).d(gz,tz)}{d(fz,gz)+d(hz,tz)}, \frac{d(fz,hz).d(gz,tz)}{d(fz,hz)+d(fz,tz)}, \frac{d(fz,tz).d(hz,tz)}{d(gz,hz)+d(gz,tz)}, \frac{d(gz,hz).d(hz,tz)}{d(fz,gz)+d(fz,hz)}\} - W(\max\{\frac{d(fz,tz).d(gz,hz)}{d(fz,hz)+d(hz,tz)}, \frac{d(fz,hz).d(gz,tz)}{d(fz,gz)+d(hz,tz)}, \frac{d(fz,hz).d(gz,tz)}{d(fz,hz)+d(fz,tz)}, \frac{d(fz,tz).d(hz,tz)}{d(gz,hz)+d(gz,tz)}, \frac{d(gz,hz).d(hz,tz)}{d(fz,gz)+d(fz,hz)}\})$$

$$d(tz,hz)$$

$$\le \max\{\frac{d(tz,tz).d(hz,hz)}{d(tz,hz)+d(hz,tz)}, \frac{d(tz,hz).d(hz,tz)}{d(tz,hz)+d(hz,tz)}, \frac{d(tz,hz).d(hz,tz)}{d(tz,hz)+d(tz,tz)}, \frac{d(tz,tz).d(hz,tz)}{d(hz,hz)+d(hz,tz)}, \frac{d(hz,hz).d(hz,tz)}{d(tz,hz)+d(tz,hz)}\} - W(\max\{\frac{d(tz,tz).d(hz,hz)}{d(tz,hz)+d(hz,tz)}, \frac{d(tz,hz).d(hz,tz)}{d(tz,hz)+d(hz,tz)}, \frac{d(tz,hz).d(hz,tz)}{d(tz,hz)+d(tz,tz)}, \frac{d(tz,tz).d(hz,tz)}{d(hz,hz)+d(hz,tz)}, \frac{d(hz,hz).d(hz,tz)}{d(tz,hz)+d(tz,hz)}\})$$

$$d(tz,hz)$$

$$\le \max\left\{0, \frac{d(tz,hz).d(hz,tz)}{2d(tz,hz)}, d(tz,hz), 0,0\right\}$$

$$- W\left(\max\left\{0, \frac{d(tz,hz).d(hz,tz)}{2d(tz,hz)}, d(tz,hz), 0,0\right\}\right)$$

$$d(tz,hz) \le d(tz,hz) - W\big(d(tz,hz)\big)$$

Which gives that $tz = hz$.

Herewehave

$$d(tfx_{2n},hgx_{2n+1}) = d(ftx_{2n},ghx_{2n+1}), \qquad \forall\, n \ge 0.$$

As $n \to \infty$, then we have $d(fz,gz) = d(hz,tz)$.

hence $fz = gz$.

From eq. (2.1),

$$d(fx_{2n},hgx_{2n+1}) = d(fx_{2n},ghx_{2n+1})$$

$$\le \max\{\frac{d(fx_{2n},tx_{2n}).d(ghx_{2n+1},hhx_{2n+1})}{d(fx_{2n},hhx_{2n+1})+d(hhx_{2n+1},tx_{2n})}, \frac{d(fx_{2n},hhx_{2n+1}).d(ghx_{2n+1},tx_{2n})}{d(fx_{2n},ghx_{2n+1})+d(hhx_{2n+1},tx_{2n})}, \frac{d(fx_{2n},hhx_{2n+1}).d(ghx_{2n+1},tx_{2n})}{d(fx_{2n},hhx_{2n+1})+d(fx_{2n},tx_{2n})}, \frac{d(fx_{2n},tx_{2n}).d(hhx_{2n+1},tx_{2n})}{d(ghx_{2n+1},hhx_{2n+1})+d(ghx_{2n+1},tx_{2n})}, \frac{d(ghx_{2n+1},hhx_{2n+1}).d(hhx_{2n+1},tx_{2n})}{d(fx_{2n},ghx_{2n+1})+d(fx_{2n},hhx_{2n+1})}\}$$

$$-W(\max\{\frac{d(fx_{2n},tx_{2n}).d(ghx_{2n+1},hhx_{2n+1})}{d(fx_{2n},hhx_{2n+1})+d(hhx_{2n+1},tx_{2n})}, \frac{d(fx_{2n},hhx_{2n+1}).d(ghx_{2n+1},tx_{2n})}{d(fx_{2n},ghx_{2n+1})+d(hhx_{2n+1},tx_{2n})}, \frac{d(fx_{2n},hhx_{2n+1}).d(ghx_{2n+1},tx_{2n})}{d(fx_{2n},hhx_{2n+1})+d(fx_{2n},tx_{2n})}, \frac{d(fx_{2n},tx_{2n}).d(hhx_{2n+1},tx_{2n})}{d(ghx_{2n+1},hhx_{2n+1})+d(ghx_{2n+1},tx_{2n})}, \frac{d(ghx_{2n+1},hhx_{2n+1}).d(hhx_{2n+1},tx_{2n})}{d(fx_{2n},ghx_{2n+1})+d(fx_{2n},hhx_{2n+1})}\})$$

As $n \to \infty$, we get

$$d(z,hz)$$

$$\le \max\left\{0, \frac{d(z,hz).d(z,hz)}{d(z,hz)+d(z,hz)}, \frac{d(z,hz).d(z,hz)}{d(z,hz)}, 0,0\right\}$$

$$- W(\max\left\{0, \frac{d(z,hz).d(z,hz)}{d(z,hz)+d(z,hz)}, \frac{d(z,hz).d(z,hz)}{d(z,hz)}, 0,0\right\}$$

$$d(z,hz) \le d(z,hz) - W\big(d(z,hz)\big)$$

Which implies that $z = hz$. Similarly we get

$$d(ffx_{2n},gx_{2n+1})$$

$$\le \max\{\frac{d(ffx_{2n},tfx_{2n}).d(gx_{2n+1},hx_{2n+1})}{d(ffx_{2n},hx_{2n+1})+d(hx_{2n+1},tfx_{2n})}, \frac{d(ffx_{2n},hx_{2n+1}).d(gx_{2n+1},tfx_{2n})}{d(ffx_{2n},gx_{2n+1})+d(hx_{2n+1},tfx_{2n})}, \frac{d(ffx_{2n},hx_{2n+1}).d(gx_{2n+1},tfx_{2n})}{d(ffx_{2n},hx_{2n+1})+d(ffx_{2n},tfx_{2n})}, \frac{d(ffx_{2n},tfx_{2n}).d(hx_{2n+1},tfx_{2n})}{d(gx_{2n+1},hx_{2n+1})+d(gx_{2n+1},tfx_{2n})}, \frac{d(gx_{2n+1},hx_{2n+1}).d(hx_{2n+1},tfx_{2n})}{d(ffx_{2n},gx_{2n+1})+d(ffx_{2n},hx_{2n+1})}\}$$

$$-W(\max\{\frac{d(ffx_{2n},tfx_{2n}).d(gx_{2n+1},hx_{2n+1})}{d(ffx_{2n},hx_{2n+1})+d(hx_{2n+1},tfx_{2n})}, \frac{d(ffx_{2n},hx_{2n+1}).d(gx_{2n+1},tfx_{2n})}{d(ffx_{2n},gx_{2n+1})+d(hx_{2n+1},tfx_{2n})}, \frac{d(ffx_{2n},hx_{2n+1}).d(gx_{2n+1},tfx_{2n})}{d(ffx_{2n},hx_{2n+1})+d(ffx_{2n},tfx_{2n})}, \frac{d(ffx_{2n},tfx_{2n}).d(hx_{2n+1},tfx_{2n})}{d(gx_{2n+1},hx_{2n+1})+d(gx_{2n+1},tfx_{2n})}, \frac{d(gx_{2n+1},hx_{2n+1}).d(hx_{2n+1},tfx_{2n})}{d(ffx_{2n},gx_{2n+1})+d(ffx_{2n},hx_{2n+1})}\}), \qquad \forall\, n \ge 0$$

As $n \to \infty$, we get

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$$\mathrm{d}(fz,z) \le \max\left\{0,\frac{\mathrm{d}(fz,z)}{2},\mathrm{d}(fz,z),0,0\right\} - W\left(\max\left\{0,\frac{\mathrm{d}(fz,z)}{2},\mathrm{d}(fz,z),0,0\right\}\right)$$

$$\mathrm{d}(fz,z) \le \mathrm{d}(fz,z) - W(\mathrm{d}(fz,z))$$

Which means that, $z = fz$.

Hence from above we have, $z = fz = gz = tz = hz$.

So z is a common fixed point of f, g, h and t.

Uniqueness - If v is another common fixed point of f, g, h and t in X

Then we have,

$$\mathrm{d}(z,v) = \mathrm{d}(fz,gv) \le \mathrm{d}(z,v) - W(\mathrm{d}(z,v)) < d(z,v)$$

Which is a contradiction. This completes the proof.

Application

Let X and Y be a Banach space ,$S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space and i_X be the identity mapping on X. $B(S)$ denotes the set of all real-valued bounded functions on S. Put

$$\mathrm{d}(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$$

It is obvious that $(B(S),\mathrm{d})$ is a complete metric space.

Define $u: S \times D \to R$, $T: S \times D \to S$ and $H_i: S \times D \times R \to R$ for $i \in \{1,2,3,4\}$.

Now we study those conditions, which is guarantee the existence and uniqueness of solution and common solution for the following system of functional equations.

$$f_i(x) = \sup_{y \in D}\left(u(x,y) + H_i\left(x,y,f_i(T(x,y))\right)\right)\} \quad \forall x \in S, i \in \{1,2,3,4\} \qquad (3.1)$$

Theorem 3.1 -If the following conditions are satisfied

(a) u and H_i are bounded for $i \in \{1,2,3,4\}$

(b) There exist a $W \in \emptyset$ and the mapping A_1, A_2 , A_3 and A_4 are defined as follows, $\forall x \in S, g_i \in B(S), i \in \{1,2,3,4\}$,

$$A_i g_i(x) = \sup_{y \in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\}$$

Satisfying

$$|H_1(x,y,g(t)) - H_2(x,y,h(t))| \le \max\Big\{\frac{d(A_1g,A_4g).d(A_2h,A_3h)}{d(A_1g,A_3h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_2h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_3h)+d(A_1g,A_4g)},\frac{d(A_1g,A_4g).d(A_3h,A_4g)}{d(A_2h,A_3h)+d(A_2h,A_4g)},\frac{d(A_2h,A_3h).d(A_3h,A_4g)}{d(A_1g,A_2h)+d(A_1g,A_3h)}\Big\}$$
$$-W(\max\Big\{\frac{d(A_1g,A_4g).d(A_2h,A_3h)}{d(A_1g,A_3h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_2h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_3h)+d(A_1g,A_4g)},\frac{d(A_1g,A_4g).d(A_3h,A_4g)}{d(A_2h,A_3h)+d(A_2h,A_4g)},\frac{d(A_2h,A_3h).d(A_3h,A_4g)}{d(A_1g,A_2h)+d(A_1g,A_3h)}\Big\})$$

for all $(x,y) \in S \times D$, $g,h \in B(S), t \in S$

(c) $A_1(B(S)) \subseteq A_3(B(S)), A_2(B(S)) \subseteq A_4(B(S))$

(d) There exist some $A_i \in \{A_1,A_2,A_3,A_4\}$ such that for any sequence $\{h_n\}_{n\ge1} \subseteq B(S)$ and $h \in B(S)$,

$$\lim_{n\to\infty}\sup_{x \in S}|h_n(x) - h(x)| = 0 \Rightarrow \lim_{n\to\infty}\sup_{x \in S}|A_ih_n(x) - A_ih(x)| = 0$$

(e) $A_1A_4 = A_4A_1$, $A_2A_3 = A_3A_2$

Then the system of functional equations (3.1) has a unique common solution in $B(S)$.

Proof – It follows from eq. (a)-(d) that A_1, A_2 , A_3 and A_4 are continuous self mapping of $B(S)$. For any $g,h \in B(S), x \in S$ and $\epsilon > 0$,there exist $y,z \in D$

$$A_1g(x) < u(x,y) + H_1\left(x,y,g(T(x,y))\right) + \epsilon \qquad (3.2)$$

$$A_2h(x) < u(x,z) + H_2\left(x,z,h(T(x,z))\right) + \epsilon \qquad (3.3)$$

Here we have

$$A_1g(x) \ge u(x,z) + H_1\left(x,z,g(T(x,z))\right) \qquad (3.4)$$

$$A_2h(x) \ge u(x,y) + H_2\left(x,y,h(T(x,y))\right) \qquad (3.5)$$

It follows from eq.(3.2),(3.5) and (b)

$$A_1g(x) - A_2h(x) < H_1\left(x,y,g(T(x,y))\right) - H_2\left(x,y,h(T(x,y))\right) + \epsilon \qquad (3.6)$$

$$\le \max\Big\{\frac{d(A_1g,A_4g).d(A_2h,A_3h)}{d(A_1g,A_3h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_2h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_3h)+d(A_1g,A_4g)},\frac{d(A_1g,A_4g).d(A_3h,A_4g)}{d(A_2h,A_3h)+d(A_2h,A_4g)},\frac{d(A_2h,A_3h).d(A_3h,A_4g)}{d(A_1g,A_2h)+d(A_1g,A_3h)}\Big\}$$
$$-W(\max\Big\{\frac{d(A_1g,A_4g).d(A_2h,A_3h)}{d(A_1g,A_3h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_2h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_3h)+d(A_1g,A_4g)},\frac{d(A_1g,A_4g).d(A_3h,A_4g)}{d(A_2h,A_3h)+d(A_2h,A_4g)},\frac{d(A_2h,A_3h).d(A_3h,A_4g)}{d(A_1g,A_2h)+d(A_1g,A_3h)}\Big\}) + \epsilon$$

From eq.(3.3),(3.4) and (b),

$$A_1g(x) - A_2h(x) > H_1\left(x,z,g(T(x,z))\right) - H_2\left(x,z,h(T(x,z))\right) - \epsilon \qquad (3.7)$$

$$\ge -\max\Big\{\frac{d(A_1g,A_4g).d(A_2h,A_3h)}{d(A_1g,A_3h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_2h)+d(A_3h,A_4g)},\frac{d(A_1g,A_3h).d(A_2h,A_4g)}{d(A_1g,A_3h)+d(A_1g,A_4g)},\frac{d(A_1g,A_4g).d(A_3h,A_4g)}{d(A_2h,A_3h)+d(A_2h,A_4g)},$$

$$\frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\}$$

$$-W(\max\{\frac{d(A_1g, A_4g).d(A_2h, A_3h)}{d(A_1g, A_3h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_2h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_3h) + d(A_1g, A_4g)}, \frac{d(A_1g, A_4g).d(A_3h, A_4g)}{d(A_2h, A_3h) + d(A_2h, A_4g)}, \frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\}) - \epsilon$$

Again from eq. (3.6) and (3.7) we get

$$d(A_1g, A_2h) = \sup_{x \in S}|A_1g(x) - A_2h(x)| \quad (3.8)$$

$$\leq \max\{\frac{d(A_1g, A_4g).d(A_2h, A_3h)}{d(A_1g, A_3h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_2h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_3h) + d(A_1g, A_4g)}, \frac{d(A_1g, A_4g).d(A_3h, A_4g)}{d(A_2h, A_3h) + d(A_2h, A_4g)}, \frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\}$$

$$-W(\max\{\frac{d(A_1g, A_4g).d(A_2h, A_3h)}{d(A_1g, A_3h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_2h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_3h) + d(A_1g, A_4g)}, \frac{d(A_1g, A_4g).d(A_3h, A_4g)}{d(A_2h, A_3h) + d(A_2h, A_4g)}, \frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\}) + \epsilon$$

Letting $\epsilon \to 0$ in eq. (3.8), we get

$$d(A_1g, A_2h) = \sup_{x \in S}|A_1g(x) - A_2h(x)| \quad (3.9)$$

$$\leq \max\{\frac{d(A_1g, A_4g).d(A_2h, A_3h)}{d(A_1g, A_3h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_2h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_3h) + d(A_1g, A_4g)}, \frac{d(A_1g, A_4g).d(A_3h, A_4g)}{d(A_2h, A_3h) + d(A_2h, A_4g)}, \frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\}$$

$$-W(\max\{\frac{d(A_1g, A_4g).d(A_2h, A_3h)}{d(A_1g, A_3h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_2h) + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h).d(A_2h, A_4g)}{d(A_1g, A_3h) + d(A_1g, A_4g)}, \frac{d(A_1g, A_4g).d(A_3h, A_4g)}{d(A_2h, A_3h) + d(A_2h, A_4g)}, \frac{d(A_2h, A_3h).d(A_3h, A_4g)}{d(A_1g, A_2h) + d(A_1g, A_3h)}\})$$

From (e) and eq. (3.9) we conclude that A_1, A_2, A_3 and A_4 have a unique common fixed point $v \in B(S)$, and $v(x)$ is a unique common solution of the system of functional equation (3.1).

This completes the proof.

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